Journal of Pure and Applied Algebra 31 (1984) 227-239
North-Holland

A DUALITY THEOREM FOR ORDERS IN CENTRAL SIMPLE ALGEBRAS OVER FUNCTION FIELDS

M. VAN DEN BERGH* and J. VAN GEEL
Dept. of Mathematics, University of Antwerp, U.I.A., Universiteitsplein 1, 2610 Wilrijk, Belgium

Communicated by F. Van Oystaeyen
Received December 1982

Introduction

Let C be a complete regular curve over a field k. If D is a central simple algebra over $K=k(C)$ and $\mathcal{O}_\Lambda$ is a sheaf of maximal $\mathcal{O}_C$-Orders in D, then it is possible to extend the Riemann–Roch theorem to a result on the arithmetic of $\mathcal{O}_\Lambda$, cf. [5]. The proof exploits very much the close relation between the arithmetic of $\mathcal{O}_\Lambda$ and $\mathcal{O}_C$ (ramification occurs only at a finite number of points). This relation is also manifested by the formula relating the genus of D to the genus of the center:

$$g_D = N^2 g_K - N^2 + 1 + \tfrac{1}{2} \sum f_p(e_p - 1),$$

where $N^2 = [D:K]$, e_p is the ramification index and f_p the residue class degree, cf. [5].

The sheaf $\mathcal{O}_\Lambda$ may be viewed as a structure sheaf of some noncommutative curve Γ in the sense of [7]. The demand that $\mathcal{O}_\Lambda$ is a sheaf of *maximal* $\mathcal{O}_C$-Orders is a regularity condition on the curve, in [7] Γ is called strongly regular. A 'good' notion of a regular curve (variety) in the noncommutative case is lacking at this time. This ties up with the fact that a theory of differentials on noncommutative varieties is missing. For commutative curves the role of differentials is made explicit in the Serre Duality theorem, which actually is an other form of the Riemann-Roch theorem.

This paper furthers on the study of noncommutative curves by proving a 'Duality theorem'.

In the first section cohomological methods are being used. (Readers who are not familiar with sheaf cohomology, are refered to Hartshorne's book [1]. We tried to follow his approach as close as possible; also our notation is compatible with his.) The advantage is that a fairly general treatment is possible. For a commutative regular scheme X over k, we consider a sheaf of $\mathcal{O}_X$-Algebras, $\mathcal{O}_\Lambda$. This $\mathcal{O}_\Lambda$ may also

* The first author is supported by an NFWO-grant.

be viewed as a sheaf on a noncommutative variety. The (technical) conditions imposed are: $\mathcal{O}_A$ has to be a flat and coherent $\mathcal{O}_X$-Module. Note that, for instance in the case X is a curve, this is more general than taking maximal $\mathcal{O}_X$-Orders; a sheaf of HNP-$\mathcal{O}_X$-Orders is also flat and coherent.

The duality theorem can be stated as follows: There exists a dualizing sheaf ω_A and a map $\eta_A : H^n(X, \omega_A) \to k$ such that for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_A$-modules the cup product followed by $\eta_A : H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_A) \to H^n(X, \omega_A) \to k$, yields an isomorphism $H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_A)^*$. Here $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_A}(\mathcal{F}, \mathcal{O}_A)$ and * denotes the dual as k-vectorspace.

The sheaf ω_A is obtained by extending the dualising sheaf of the central scheme in an obvious way, namely $\omega_A = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_A, \omega_X)$. The map η_A is the composition of η_X with a trace map $\mathrm{Tr} : \omega_A \to \omega_X$. The section ends with an interpretation of ω_A and η_A. In case X is a curve, ω_A is related to the different of the central simple algebra D and the trace map Tr is induced by the reduced trace of D.

In Section 2 the duality theorem for strongly regular noncommutative curves is obtained in terms of repartitions R as defined in [5] and 'differentials'. The algebra of 'differentials' used here is obtained from the differentials of the central curve by tensoring up with D: $\Omega_{D/k} = \Omega_{K/k} \otimes_K D$. We define the residue map on $\Omega_{D/k}$ in a point P, Res_P, as the composition of the classical residue map and the reduced trace. The map $\sum \mathrm{Res}_P$ then defines an inproduct between $\Omega_{D/k}$ and R. Duality between certain subspaces of $\Omega_{D/k}$ and k-linear forms on R (valuation forms) is obtained in the obvious way from this inproduct.

Section 3 explains the relation between both forms of the duality theorem. The Riemann–Roch theorem is reconsidered. The notion of canonical divisor is discussed; it turns out to be a divisor associated in a natural way to differentials. Also the genus is interpreted in terms of dimensions of spaces of differentials and in terms of the dimensions of the cohomology spaces $H^0(X, \mathcal{O}_A)$ and $H^1(X, \mathcal{O}_A)$.

In the Appendix we show that the residue map can be defined without using the residue map of the center. This is done by using Trace maps as studied by Tate in [4]. We also give a method to calculate the residues.

The results we obtained are not essentially new. The classical results on duality generalize in a straightforward way to the noncommutative cases discussed. Again this is due to the relation between $\mathcal{O}_A$ and the central sheaf $\mathcal{O}_X$, a situation exploited by the extensive use of the reduced trace in different forms.

We think the benefit of this paper, expliciting what has been known in spirit, is in scaffolding for the further study of noncommutative varieties and of orders in central algebras over function fields.

1. Cohomological approach

Let X be a projective Cohen–Macauly scheme of dimension n over a field k. $\mathcal{O}_A$ is a sheaf of $\mathcal{O}_X$-Algebras which is coherent and flat as $\mathcal{O}_X$-Module. The term coherent

$\mathscr{O}_A$-Module will be used for $\mathscr{O}_A$-Modules $\mathscr{F}$ which are coherent as $\mathscr{O}_X$-Modules.

Recall the classical result on duality (Serre–Grothendieck): There exists a dualizing sheaf ω_X and a map $\eta_X : H^n(X, \omega_X) \to k$, such that for any coherent $\mathscr{O}_X$-Module $\mathscr{F}$ the Yoneda pairing followed by η_X

$$H^{n-p}(X, \mathscr{F}) \times \operatorname{Ext}^p_{\mathscr{O}_X}(\mathscr{F}, \omega_X) \to H^n(X, \omega_X) \to k \tag{1}$$

yields an isomorphism: $\operatorname{Ext}^{n-p}_{\mathscr{O}_X}(\mathscr{F}, \omega_X) \simeq H^p(X, \mathscr{F})^*$, for $0 \le p \le n$. * denotes the dual as k-space.

A sequence (1) satisfying this property is called nonsingular, cf. [1].

We define $\omega_A = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_A, \omega_X)$. There exists a canonical map $\mathrm{Tr} : \omega_A \to \omega_X$ defined by $\mathrm{Tr}(U)(f) = f(1)$, for all open sets U in X. Furthermore for every coherent $\mathscr{O}_A$-Module $\mathscr{F}$ the following natural isomorphisms hold:

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_X) \simeq \operatorname{Hom}_{\mathscr{O}_A}(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_A, \omega_X)) \simeq \operatorname{Hom}_{\mathscr{O}_A}(\mathscr{F}, \omega_A). \tag{2}$$

Now if η_A is defined as follows: $\eta_A = \eta_X \circ H^n(X, \mathrm{Tr})$, i.e. η_A maps $H^n(X, \omega_A)$ to k, then the following holds:

Theorem 1. *The sequence defined by the Yoneda pairing and η_A:*

$$H^{n-p}(X, \mathscr{F}) \times \operatorname{Ext}^p_{\mathscr{O}_A}(\mathscr{F}, \omega_A) \to H^n(X, \omega_A) \to k \tag{3}$$

yields an isomorphism: $\operatorname{Ext}^{n-p}_{\mathscr{O}_A}(\mathscr{F}, \omega_A) \simeq H^p(X, \mathscr{F})^*$, *for* $0 \le p \le n$.

The pair $(\eta_A \omega_A)$ is defined up to isomorphism by this property.

Proof. Let $0 \to \omega_X \to \mathscr{I}^0 \to \cdots \to \mathscr{I}^i \to \cdots$ be an injective resolution of ω_X. Since $\mathscr{O}_A$ is flat and coherent it follows that the sequence

$$0 \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_A, \omega_X) \to \cdots \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_A, \mathscr{I}^i) \to \cdots$$

is exact. Furthermore from (2) it is clear that $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_A, \mathscr{I}^i)$ is injective. Now apply the functor $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \ )$ to the first resolution and $\operatorname{Hom}_{\mathscr{O}_A}(\mathscr{F}, \ )$ to the second one. Then, again using (2):

$$\operatorname{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \omega_X) \simeq \operatorname{Ext}^i_{\mathscr{O}_A}(\mathscr{F}, \omega_A). \tag{4}$$

Consider the diagram:

$$\begin{tikzcd}
H^{n-p}(X, \mathscr{F}) \times \operatorname{Ext}^p_{\mathscr{O}_X}(\mathscr{F}, \omega_X) \arrow[r] & H^n(X, \omega_X) \arrow[r] & k \\
H^{n-p}(X, \mathscr{F}) \times \operatorname{Ext}^p_{\mathscr{O}_A}(\mathscr{F}, \omega_A) \arrow[r] \arrow[u, "\simeq"] & H^n(X, \omega_A) \arrow[r] \arrow[u, "H^n(X,\mathrm{Tr})"] & k \arrow[u, equal]
\end{tikzcd} \tag{5}$$

A straightforward calculation shows that this is a commutative diagram. The fact that the upper pairing is nonsingular implies that the lower one is nonsingular too. The proof of the uniqueness of (η_A, ω_A) follows in the same way as in the classical case, cf. [1]. $\square$

Corollary 2 (Duality I). *Let $\mathcal{F}$ be a coherent projective $\mathcal{O}_\Lambda$-Module, define $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_\Lambda}(\mathcal{F}, \mathcal{O}_\Lambda)$. Then there exists a natural isomorphism:*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_\Lambda)^*. \tag{6}$$

Proof. This follows directly from Theorem 1 and the following isomorphisms:

$$\mathrm{Ext}^i_{\mathcal{O}_\Lambda}(\mathcal{O}_\Lambda \otimes \mathcal{F}, \omega_\Lambda) \simeq \mathrm{Ext}^i_{\mathcal{O}_\Lambda}(\mathcal{O}_\Lambda, \mathcal{F}^\vee \otimes \omega_\Lambda) \quad \text{and} \quad \mathrm{Ext}^i_{\mathcal{O}_\Lambda}(\mathcal{O}_\Lambda, \mathcal{G}) \simeq H^i(X, \mathcal{G})$$

for any $\mathcal{O}_\Lambda$-Module $\mathcal{G}$, cf. [1]. □

Interpretation

In order to obtain some interpretation of the sheaf ω_Λ (the dualizing sheaf) and of the map η_Λ, we specify X and $\mathcal{O}_\Lambda$ as follows: X stands for a regular integral scheme and $\mathcal{O}_\Lambda = \mathcal{O}_\Lambda$ will be an $\mathcal{O}_X$-Order. By the latter we mean the following: $\Gamma(U, \mathcal{O}_\Lambda)$ is a $\Gamma(U, \mathcal{O}_X)$-order in a fixed central simple algebra D for all U affine in X. The central simple algebra D is obtained as $\mathcal{O}_{\Lambda,\gamma}$, where γ is the generic point of X, its center is $K = \mathcal{O}_{X,\gamma}$ the function field of X. In the sequel we fix some imbedding of $\mathcal{O}_X$ and $\mathcal{O}_\Lambda$ in the constant sheaves $\boldsymbol{K}$ and $\boldsymbol{D}$ over X.

Since ω_X is locally free of rank 1, it may be viewed (in a noncanonical way) as a subsheaf of $\boldsymbol{K}$. Let $\mathbf{tr}: \boldsymbol{D} \to \boldsymbol{K}$ be defined by the reduced trace map of central simple algebras as follows: $\mathbf{tr}(U)(x) = \mathrm{tr}(x)$ and consider $\boldsymbol{V}_\Lambda : U \to \{x \in \Gamma(U, \boldsymbol{D}) \mid$ for all V open in U, $\mathrm{tr}(x\Gamma(V, \mathcal{O}_\Lambda))$ is a subset of $\Gamma(U, \omega_X)\}$. It is easy to verify that $\boldsymbol{V}_\Lambda$ is a sheaf. The map $\theta : \boldsymbol{V}_\Lambda \to \omega_\Lambda$; $x \to \mathbf{tr}(x-)$ defines an isomorphism on the stalks of $\boldsymbol{V}_\Lambda$ and ω_Λ, because the reduced trace map of central simple algebras generates all K-linear maps from D to K, moreover $\mathrm{tr}(xy)$ defines a nondegenerated bilinear form. So

$$\boldsymbol{V}_\Lambda \simeq \omega_\Lambda. \tag{7}$$

This allows us to imbed ω_Λ in $\boldsymbol{D}$. The map $\mathrm{Tr}: \omega_\Lambda \to \omega_X$ defined before is nothing but the restriction of $\mathbf{tr}: \boldsymbol{D} \to \boldsymbol{K}$ to ω_Λ via this imbedding.

If $X = C$ is of dimension 1, i.e. if it is a curve, ω_C is the sheaf of differentials on C. Let $\mathrm{Res}_{K,x}$ be the residue map from $\omega_K = \Gamma(C, \boldsymbol{K} \otimes \omega_C)$ to k. The composition of this map with the noncanonical identification of ω_K with K (ω_K is locally free of rank 1), yields a map from $K \to k$ also denoted $\mathrm{Res}_{K,x}$.

The residue theorem tells us that for all a in K, $\sum_x \mathrm{Res}_{K,x}(a) = 0$. We define $\mathrm{Res}_{D,x} : D \to k$ by $\mathrm{Res}_{D,x}(u) = \mathrm{Res}_{K,x}(\mathrm{tr}\, u)$ for all u in D. So it follows immediately that $\sum_x \mathrm{Res}_{D,x}(u) = 0$ for all u in D. Consider the diagram, cf. [1]:

$$\begin{tikzcd}
0 \arrow[r] & \omega_C \arrow[r] & \boldsymbol{K} \arrow[r] & \bigoplus_{x \in C} i_*(\boldsymbol{K}_x/\omega_{C,x}) \arrow[r] & 0 \\
0 \arrow[r] & \omega_\Lambda \arrow[r] \arrow[u, "\mathrm{Tr}"'] & \boldsymbol{D} \arrow[r] \arrow[u, "\mathbf{tr}"'] & \bigoplus_{x \in C} i_*(\boldsymbol{D}_x/\omega_{\Lambda,x}) \arrow[r] \arrow[u, "\oplus\mathrm{tr}"'] & 0
\end{tikzcd} \tag{8}$$

Taking cohomology yields:

$$\begin{tikzcd} K \arrow[r] & \bigoplus K/\omega_{C,x} \arrow[r] & H^1(C,\omega_C) \arrow[r] & 0 \\ D \arrow[r] \arrow[u] & \bigoplus D/\omega_{\Lambda,x} \arrow[r] \arrow[u] & H^1(C,\omega_\Lambda) \arrow[r] \arrow[u, "H^1(C,\mathrm{tr})"'] & 0 \end{tikzcd} \tag{9}$$

Define a map $\mathrm{Res}: \bigoplus D/\omega_{\Lambda,x} \to k$: $(u_x) \to \sum \mathrm{Res}_{D,x}(u_x)$. As noted above this map vanishes on D hence it passes to the quotient and yields a map t: $H^1(C,\omega_\Lambda) \to k$. It follows from Theorem 1 that this map t is the 'same' as the map η_Λ.

2. A direct approach (in the case of curves)

Let C be a complete regular curve over some field k, let $K = k(C)$ be its function field. There is a one-to-one correspondence between the points p of C and the k-valuation rings of K. This makes the study of the curve C equivalent to the study of the extension K/k. In the sequel we use the same symbol p for a point on C as well as for the maximal ideal of the corresponding valuation ring O_p.

If D is a central simple K-algebra, we define a *noncommutative curve* with function algebra D as follows: For every point p on C we choose one maximal O_p-order Λ_p in D. The theory of maximal orders over discrete valuation rings tells us that Λ_p has a unique maximal ideal P. The set of all these ideals P, say Γ is called an *algebraic model of a noncommutative curve* in D.

Recall some facts about maximal O_p-orders, cf. [2]. A maximal order Λ_p over a discrete valuation ring O_p is a principal ideal ring. Its unique maximal ideal P is of the form: $P = T\Lambda_p = \Lambda_p T$ for some T in D. Every twosided ideal of Λ_p is a power of P. It follows that $p\Lambda_p = P^{e_p}$. If $e_p = 1$, p is said to be *unramified* in D, otherwise p is *ramified*. The number e_p is called the *ramification index* of p. Note also that only finitely many primes p are ramified in D. The quotient Λ_p/P is called the *residue algebra* of Λ_p, it is finite dimensional over $O_p/p = k_p$ the residue field of p. If $[\Lambda_p/P : k_p] = \Psi_p$, the *relative residue class degree* of P, then $\Psi_p e_p = N^2$ the degree of D over K. The number $[\Lambda_p/P : k] = f_p$ is called the absolute *residue class degree* of P.

Divisors, repartitions (idèles), valuation forms

An element δ in the free abelian group with basis the elements of Γ is a *divisor* on Γ, i.e. $\delta = \sum_{P\in\Gamma} n_P P$ with almost all $n_P = 0$. The coefficients n_P of a divisor δ are denoted by $\mathrm{ord}_P(\delta)$. The *degree of a divisor* δ is $\deg(\delta) = \sum_{P\in\Gamma} f_p \,\mathrm{ord}_P(\delta)$.

It is possible to associate with every element of D a divisor. If r is in D, then $\Lambda_p r \Lambda_p = P^{v_P(r)}$ with $v_P(r)$ in $\mathbb{Z}$; this follows from the fact that $\Lambda_p r \Lambda_p$ is a fractional ideal. We define the divisor $(r) = \sum v_P(r) P$, cf. [5].

A *repartition* is a family $\{r_P\}_{P\in\Gamma}$ of elements in D such that $r_P \in \Lambda_P$ for almost all P. The set of all repartitions R forms a k-algebra.

If δ is a divisor on Γ consider the sub-k-algebra

$$R(\delta) = \{\{r_P\} \mid \upsilon_P(r_P) \geq -\mathrm{ord}_P(\delta) \text{ for all } P \text{ in } \Gamma\}.$$

A *valuation form* is a linear form on the k-vectorspace $R/R(\delta)+D$, i.e. an element of the dual space of R vanishing on $R(\delta)+D$, for some divisor δ on Γ.

Definition 3. Let $\Omega_{K/k}$ be the algebra of differentials on K, then by analogy we call $\Omega_{D/k} = \Omega_{K/k} \otimes_K D$ the *algebra of differentials on D*.

Since $\Omega_{K/k}$ is one-dimensional over K, every element of $\Omega_{D/k}$ may be written as $r \otimes \mathrm{d}g = r\,\mathrm{d}g$ for some fixed differential $\mathrm{d}g$ in $\Omega_{K/k}$. If $\mathrm{Res}_{K,p}$ is the residue map defined on $\Omega_{K/k}$ in p, then we define $\mathrm{Res}_{D,P}\colon \Omega_{D/k} \to k$ to be the composition of $\mathrm{Res}_{K,p}$ and the reduced trace map on D, i.e.

$$\mathrm{Res}_{D,P}(r\,\mathrm{d}g) = \mathrm{Res}_{K,p}(\mathrm{tr}(r)\,\mathrm{d}g).$$

Note that the restriction of $\mathrm{Res}_{D,P}$ is not $\mathrm{Res}_{K,p}$ but $N^2\,\mathrm{Res}_{K,p}$. The residue theorem follows from the commutative version, immediately from this definition (in the same was as in Section 1).

Let t_p be a generator of p in O_p, then every element of $\Omega_{D/k}$ may be written as $r\,\mathrm{d}t_p$ for some r in D. We define

$$\upsilon_P(r\,\mathrm{d}t_p) = e_p - 1 + \upsilon_P(r). \tag{10}$$

Since a different choice of t_p will change r by a unit element in O_p. this definition does not depend on the choice of t_p.

For every element α in $\Omega_{D/k}$ we consider the divisor $(\alpha)_D = \sum \upsilon_P(\alpha)P$. The set $\Omega(\delta) = \{\alpha \mid (\alpha)_D \geq \delta\}$ is a k-subspace of $\Omega_{D/k}$ associated to a divisor δ on Γ.

The residue map defines an inproduct between elements of $\Omega_{D/k}$, differentials, and elements of R, repartitions on Γ. For α in $\Omega_{D/k}$, and $\{r_P\}_P$ in R we have:

$$\langle \alpha, \{r_P\}_P \rangle = \sum \mathrm{Res}_{D,P}(r_P\alpha). \tag{11}$$

The following properties hold:

(1) $\langle \alpha, \{r_P\}_P\rangle = 0$ if $\{r_P\}_P$ is in $R(\delta)$ and α is an element of $\Omega(\delta)$. This follows from the fact that $\upsilon_P(r_P\alpha) \geq 0$ for all P. So if $r_P\alpha = s\,\mathrm{d}t_p$, then $\upsilon_P(s) \geq 0$, i.e. s is in Λ_p and therefore $\mathrm{tr}(s)$ is in O_p. The result follows from the fact that $\mathrm{Res}_{K,p}$ vanishes on O_p.

(2) $\langle \alpha, \{r_P\}_P\rangle = 0$ if $r_P = r$ for all P. This follows from the residue theorem.

(3) For s in D; $\langle s\alpha, \{r_P\}_P\rangle = \langle \alpha, \{sr_P\}_P\rangle$. This inproduct defines in the usual way a map

$$\Theta : \Omega_{D/k} \to R^*\colon \quad \alpha \to \langle \alpha, - \rangle.$$

The set of valuation forms $(R/R(\delta)+D)^*$ is denoted $J(\delta)$.

Lemma 4. *If α is an element of $\Omega_{D/k}$ such that $\Theta(\alpha)$ is in $J(\delta)$, then α is an element of $\Omega(\delta)$.*

Proof. If α is not in $\Omega(\delta)$, then for some v_P, $v_P(\alpha) < \mathrm{ord}_P(\delta)$. Put $n = v_P(\alpha)+1$. Then $\alpha = s\,\mathrm{d}t_p$ with $s = uT_P^{n-e}p$. Consider the repartition:

$$\begin{cases} r_Q = 0 & \text{if } Q \neq P, \\ r_P = u'T_P^{-n}. \end{cases}$$

Then $r_P\alpha = uT_P^{-n}u'T_P^{n-e}p\,\mathrm{d}t_p = uu''T_P^{-e}p\,\mathrm{d}t_p$. The last term of these equalities is equal to $t_p^{-1}\,\mathrm{d}t_p$ if we choose u' properly. Since the residue of $t_p^{-1}\,\mathrm{d}t_p$ is not zero, also $\langle \alpha, \{r_Q\}_Q\rangle \neq 0$. But $n \leq \mathrm{ord}_P(\delta)$ so $\{r_Q\}_Q$ is in $R(\delta)$, contradicting the fact that $\Theta(\alpha)$ is zero on $R(\delta)$. □

Remark. The proof of Lemma 4 in the above form only holds if $\mathrm{char}(k)$ does not divide N^2. Otherwise $\mathrm{Res}_{D,P}\,t_p^{-1}\,\mathrm{d}t_p = N^2\,\mathrm{Res}_{K,P}\,t_p^{-1}\,\mathrm{d}t_p = 0$. However we will show in the Appendix that for a suitable choice of T_P, $\mathrm{Res}_{D/k}\,T_P^{-e}\,\mathrm{d}t_p \neq 0$ even in the case $\mathrm{char}(k) \mid N^2$.

Theorem 5 (Duality II). *For every divisor δ on Γ, the map Θ defines an isomorphism between $\Omega(\delta)$ and $J(\delta)$.*

Proof. Injectivity follows from Lemma 4.

Θ is surjective on $\Omega_{D/k}$ since both $\Omega_{D/k}$ and R^* are N^2-dimensional over K, cf. [5]. Surjectivity on $\Omega(\delta)$ then follows again from Lemma 4. □

3. Duality I and II compared

Let C be again a complete regular curve over k, $\mathcal{O}_C$ its structure sheaf. A sheaf of maximal $\mathcal{O}_C$-Orders $\mathcal{O}_\Lambda$ defines a noncommutative curve Γ in an obvious way.

Without going into full detail, we would like to point out the relation between both forms of the 'Duality theorem'.

Since C is a complete regular curve the dualising sheaf ω_C is exactly the sheaf of differentials on C, cf. [1]. So $\boldsymbol{K} \otimes_{\mathcal{O}_C} \omega_C = \omega_{K/k}$ and $\Gamma(C, \omega_{K/k}) = \Omega_{K/k}$ the algebra of differentials on K.

Let $\omega_{D/k} = \boldsymbol{K} \otimes_{\mathcal{O}_C} \omega_\Lambda$ and $\Omega_{D/k} = \Gamma(C, \omega_{D/k})$. First note that this definition of differentials on D is compatible with Definition 3, because

$$\begin{aligned} \omega_{D/k} &= \boldsymbol{K} \otimes_{\mathcal{O}_C} \omega_\Lambda \simeq \mathscr{H}om_{\boldsymbol{K}}(\boldsymbol{D}, \omega_{K/k}) \\ &\simeq \mathscr{H}om_{\boldsymbol{K}}(\boldsymbol{D}, \boldsymbol{K}) \otimes_{\boldsymbol{K}} \omega_{K/k} \simeq \boldsymbol{D} \otimes_{\boldsymbol{K}} \omega_{K/k}. \end{aligned} \tag{12}$$

The isomorphism $\mathscr{H}om_{\boldsymbol{K}}(\boldsymbol{D}, \boldsymbol{K}) \simeq \boldsymbol{D}$ is obtained via the reduced trace map. Recall that $\omega_\Lambda \simeq \mathcal{V}_\Lambda$, cf. formula (7). $\mathcal{V}_\Lambda \simeq \Delta_\Lambda^{-1} \otimes_{\mathcal{O}_C} \omega_C$, where Δ_Λ^{-1} is the inverse different sheaf of D, i.e.

$$\Delta_\Lambda^{-1}(U) = \{r \in \Gamma(U,D) \mid \text{for all open } V \text{ in } U,\ \operatorname{tr}(r\Gamma(V,\mathcal{O}_\Lambda)) \text{ is a subset of } \Gamma(V,\mathcal{O}_C)\}. \tag{13}$$

It follows from the theory of maximal orders that Δ_Λ^{-1} is invertible and

$$\Delta_\Lambda(U) = \{r \in \Gamma(U,D) \mid r \in \mathcal{P}_{\Lambda,p}^{e_p-1} \text{ for all } p \text{ in } C\}, \tag{14}$$

$\mathcal{P}_{\Lambda,p} = P$ is the maximal ideal of $\mathcal{O}_{\Lambda,p}$.

We have the following commutative diagram:

$$\begin{tikzcd}
\omega_\Lambda \simeq \Delta_\Lambda^{-1} \otimes_C \omega_C \arrow[r] & D \otimes_K \omega_{K/k} \simeq \omega_{D/k} \arrow[r] & D \\
\omega_C \arrow[u] \arrow[r] & \omega_{K/k} \arrow[u] \arrow[r] & K \arrow[u]
\end{tikzcd} \tag{15}$$

The vertical arrows being the canonical imbeddings. (Note that the resulting imbedding of ω_Λ in D is the same as the one we obtained earlier in this paper.)

For every divisor δ on Γ it is possible to construct an invertible subsheaf of D, $\mathcal{L}(\delta)$. Its inverse is given by its dual $\mathcal{L}(\delta)^v$.

$$\mathcal{L}(\delta)(U) = \{r \in \Gamma(U,D) \mid r \in \mathcal{P}_{\Lambda,p}^{-\operatorname{ord}_P \delta} \text{ for all } p \text{ in } C\}. \tag{16}$$

Actually this yields a one-to-one correspondence between divisors and invertible subsheafs of D, which is easy to check. Therefore the imbeddings $\omega_C \to K$, $\omega_\Lambda \to D$ and $\Delta_\Lambda \to D$ correspond to divisors ϕ on C, ψ and Δ on Γ (Δ is the difference of $\mathcal{O}_\Lambda$, cf. [2]). The following formula follows from the diagram (15):

$$\operatorname{ord}_P \psi = e_p \operatorname{ord}_p \phi + \operatorname{ord}_P \Delta = e_p \operatorname{ord}_p \phi + e_p - 1. \tag{16}$$

The p-adic valuation on K and the P-adic pseudo-valuation on D induce 'valuations' on $\Omega_{K/k}$ and $\Omega_{D/k}$:

$$\begin{aligned}
v_p &: \Omega_{K/k} \to \mathbb{Z}: \quad a \mapsto v_p(j_p(a_p)) - \operatorname{ord}_p \phi, \\
v_P &: \Omega_{D/k} \to \mathbb{Z}: \quad \alpha \mapsto v_P(j_p'(\alpha_p)) - \operatorname{ord}_P \psi
\end{aligned} \tag{17}$$

where j_p, j_p' are the horizontal maps of the right square of diagram (15). Let t_p be a generator of the maximal ideal p in O_p. Every element of $\Omega_{D/k}$ may be written as $r\,\mathrm{d}t_p$ for some r in D. We obtain:

$$\begin{aligned}
v_P(r\,\mathrm{d}t_p) &= v_P(j_p'(r\,\mathrm{d}t_p)) - \operatorname{ord}_P \psi \\
&= v_P(r) + v_P(j_p'\mathrm{d}t_p) - \operatorname{ord}_P \psi \\
&= v_P(r) + e_p v_p(j_p\,\mathrm{d}t_p) - e_p \operatorname{ord}_p \phi - 1 \\
&= v_P(r) + e_p - 1.
\end{aligned}$$

This shows that the definition of υ_P given here is the same as the one given in the previous section.

Let δ be a divisor on D. Consider the exact sequence:

$$0 \to \mathscr{L}(\delta) \to D \to \bigoplus D_p/\mathscr{L}(\delta)_p \to 0. \tag{18}$$

Taking cohomology yields:

$$H^1(C, \mathscr{L}(\delta)) = \mathrm{Coker}(D \to \bigoplus D_p/\mathscr{L}(\delta)_p).$$

The k-space $\bigoplus D_p/\mathscr{L}(\delta)_p$ is clearly isomorphic to $R/R(\delta)$, so

$$H^1(C, \mathscr{L}(\delta)) \simeq R/R(\delta) + D. \tag{19}$$

Rest to calculate $H^0(C, \mathscr{L}^\upsilon(\delta) \otimes \omega_\Lambda) = \Gamma(C, \mathscr{L}^\upsilon(\delta) \otimes \omega_\Lambda)$. We have $\mathscr{L}^\upsilon(\delta) \simeq \mathscr{L}(-\delta)$. Since $\mathscr{L}(-\delta) \otimes \omega_\Lambda$ is imbeddable in $D \otimes \omega_{D/k} = \omega_{D/k}$, we may view the elements of $\Gamma(C, \mathscr{L}(-\delta) \otimes \omega_\Lambda)$ as elements of $\Gamma(C, \omega_{D/k}) = \Omega_{D/k}$.

An element $\alpha \in \Omega_{D/k}$ is in $H^0(C, \mathscr{L}(-\delta) \otimes \omega_\Lambda)$ if and only if for all p in C, α_p is in $\mathscr{L}(-\delta)_{p^{\omega_{\Lambda,p}}} = \mathscr{P}_p^{\mathrm{ord}_p(\delta)} \omega_{\Lambda,p}$, which is equivalent with $\upsilon_P(\alpha_p) \geq \mathrm{ord}_p(\delta)$ for all p in C. We obtained:

$$H^0(C, \mathscr{L}^\upsilon(\delta) \otimes \omega_\Lambda) \simeq \Omega(\delta). \tag{20}$$

Formulas (19) and (20) show that the Duality theorems I and II are the same.

Riemann–Roch and canonical divisors

Let C, Γ be as before. Let δ be a divisor on Γ, consider the k-vectorspace:

$$L(\delta) = H^0(C, \mathscr{L}(\delta)) = \{r \in D \mid \upsilon_P(r) \geq -\delta \text{ for all } P \in \Gamma\}.$$

Denote $\dim_k L(\delta) = l(\delta)$. In [5] the second author proved that $l(\delta)$ is finite and that:

$$l(\delta) = \deg \delta + 1 - g_D + \dim_k R/R(\delta) + D. \tag{21}$$

Where the number g_D depends only on the algebra D, it is called the *genus* of D. From the Duality theorem follows that

$$\dim_k R/R(\delta) + D = \dim_k \Omega(\delta) = \dim_k (H^1(C, \mathscr{L}(\delta)).$$

Let $\mathrm{d}s$ be a fixed differential in $\Omega_{K/k}$. Then every element of $\Omega_{D/k}$ is of the form $r\,\mathrm{d}s$, with r in D. Furthermore $r\,\mathrm{d}s$ is in $\Omega(\delta)$ iff $(r\,\mathrm{d}s)_D \geq \delta$, i.e. iff $\upsilon_P(r\,\mathrm{d}s) \geq \mathrm{ord}_P \delta$ for all P in Γ. Therefore $r\,\mathrm{d}s$ is in $\Omega(\delta)$ iff r is in $L((\mathrm{d}s)_D - \delta)$.

The divisor $(\mathrm{d}s)_D$ is called a *canonical divisor*. Since $\mathrm{d}s$ is in $\Omega_{K/k}$ it also induces a divisor on C, $(\mathrm{d}s)_K$. The relation between $(\mathrm{d}s)_K$ and $(\mathrm{d}s)_D$ follows from the definition of υ_P; we obtain

$$\mathrm{ord}_P(\mathrm{d}s)_D = e_p \,\mathrm{ord}_p(\mathrm{d}s)_K - 1. \tag{22}$$

So a canonical divisor κ is obtained from a canonical divisor of the center added to the different Δ. And

$$l(\delta) = \deg \delta + 1 - g_D + l(\kappa - \delta) \tag{23}$$

This formula was also obtained in [5], after some rather lengthy calculations. Here it follows more elegantly from the duality theorem, moreover the notion of a canonical divisor is obtained in a more natural way as divisors of 'differentials'.

Remark. If ζ is the zero divisor, i.e. $\mathrm{ord}_P(\zeta) = 0$ for all P, then $L(\zeta) = \bigcap_p \Lambda_p$ consists of k-algebraic elements in D.

In the case D is a skew field it is not difficult to prove that $L(\zeta)$ is a skew field too. Its k-dimension is given by the formula

$$l(\zeta) = 1 - g_D + l(\kappa). \tag{24}$$

It is tempting, in view of the intrinsic definiton of κ and the fact that g_D does not depend on the model Γ (i.e. on the choice of the maximal O_p-orders), to conjecture that $l(\zeta)$ is invariant of the model Γ. However this is not true; in [6] some counter-examples will be discussed.

Finally let us give some interpretations of the genus g_D. Put $l(\zeta) = n_\Gamma$, $g_\Gamma = g_D - 1 + n_\Gamma$ (the *relative genus* of Γ). Applying (23) to the divisors ζ and κ yields:

$$\begin{aligned} \deg \kappa &= 2g_D - 2 = 2g_\Gamma - 2n_\Gamma, \\ l(\kappa) &= g_\Gamma = g_D - 1 + n_\Gamma. \end{aligned} \tag{25}$$

As in the classical case we have

$$\begin{aligned} g_\Gamma &= \dim_k(\Omega(\zeta)) = \dim_k H^1(C, \mathcal{O}_\Lambda), \\ g_D - 1 &= \dim_k H^1(C, \mathcal{O}_\Lambda) - \dim_k H^0(C, \mathcal{O}_\Lambda). \end{aligned} \tag{26}$$

Appendix

In this appendix we will give a direct way to define the residue map on $\Omega_{D/k} = \Omega_{K/k} \otimes_K D$. The definition is based on Tate's results on trace maps and residues.

We recall from [4]: Let k be a fixed groundfield and V a k-vectorspace. An endomorphism θ of V is finite potent if $\theta^n V$ is finite-dimensional for some n.

Lemma A1. *For finite potent elements θ in* $\mathrm{End}(V)$ *a trace map* $\mathrm{Tr}_V(\theta)$ *exists.* Tr_V *has the usual properties of trace operators.*

Furthermore if W is a subspace of V and θW is contained in W, then $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta) + \mathrm{Tr}_{V/W}(\theta)$. *Traces of nilpotent elements are zero.*

Proof. Cf. [4, Section 1]. □

Let A be a k-algebra, V an A-module and W a k-subspace of V such that $fW+W/W$ is finite-dimensional for every f in A; notation $fW<W$. Define E, E_0, E_1, E_2 subspaces of $\mathrm{End}(V)$ as follows: $\theta \in E$ iff $\theta W<W$; $\theta \in E_1$ iff $\theta V<W$; $\theta \in E_2$ iff $\theta W<(0)$; $\theta \in E_0$ iff $\theta V<W$ and $\theta W<(0)$.

Lemma A2. *E is a k-subalgebra of* $\mathrm{End}(V)$*, the E_i's are twosided ideals in E; $E_1 \cap E_2 = E_0$, $E_1+E_2=E$ and every element of E_0 is finite potent. If $\phi \in E_1$ and $\psi \in E_1$, then the commutator $[\phi, \psi]$ is finite potent, so* $\mathrm{Tr}_V([\omega, \psi])$ *is defined. In case either ϕ or ψ is in E_2,* $\mathrm{Tr}_V([\phi,\psi])=0$.

Proof. Cf. [4]. □

Since A operates on V through E every element f in A may be decomposed in $f=f_1+f_2$, f_1 in E_1 and f_2 in E_2. So let f, g be in A such that $fg=gf$ (e.g. f or g in the center of A), $f \equiv f_1 \pmod{E_1}$ and $f \equiv f_2 \pmod{E_2}$. Then $\mathrm{Tr}_V([f_1, g_1])$ is defined and depends only on f and g (use Lemma A2).

Let A be a commutative algebra, let $\Omega_{A/k}$ denote the algebra of differentials of the first kind on A, the map $\mathrm{Res}_W^V(f\,dg)=\mathrm{Tr}_V([f_1, g_1])$ is a well-defined k-linear map from $\Omega_{A/k} \to k$, cf. [4, Theorem 1].

Consider again a noncommutative curve Γ over k. Let D be its function algebra. If P is a point on Γ, then for every element r in D there is an n in $\mathbb{Z}$ such that $\Lambda_p r \Lambda_p = T_P^n \Lambda_p$, it follows that $r\Lambda_p + \Lambda_p/\Lambda_p$ is finite-dimensional over k.

If we take $V=D$, $W=\Lambda_p$, then with the above notions it follows that every r in D is in E. Therefore we have for every pair r, s of elements in D a trace map $\mathrm{Tr}_P([r_1, s_1])$, where r_1, s_1 are the components of r and s in E_1.

Proposition A3. *Let* $\mathrm{tr}: D \to K$ *be the reduced trace of D. If r is in D and s in K, then* $\mathrm{Tr}_P([r_1, s_1]) = \mathrm{Tr}_p([\mathrm{tr}(r)_1, s_1])$*, where* Tr_p *is defined with respect to $V=K$, $W=O_p$.*

Proof. It is enough to prove the result for the full matrix algebra $M_n(K)$, the general case is reduced to this one by considering some representation of D in $M_n(L)$. This works out all right by the definition of the reduced trace of a central simple algebra.

Let r be in $M_n(K)$, $r=(r_{ij})$ with r_{ij} in K. If u: $K \to O_p$ is a k-linear projection and u' is defined by $u'(r)=(u(r_{ij}))$, then u' is a k-linear projection of $M_n(K) \to \Lambda_p$ (Λ_p may be taken $M_n(O_p)$ by chosing a proper representation of D). So $u'(r)$ is in E_1, $[u'(r), s_1]$ is a matrix over K with entries $[u'(r), s_1]_{ij} = [u(r_{ij}), s_1]$. Then $\mathrm{Tr}([u'(r), s_1]_{ij}) = \sum \mathrm{Tr}[u'(r), s_1]_{ii}$; to see this decompose a matrix in a diagonal one plus two triangular (=nilpotent) matrices. Using linearity of Tr and tr we find the desired result. □

We define the residue of a differential on D, i.e. an element of $\Omega_{D/k}$, in a point P of Γ as follows:

$$\mathrm{Res}_P r\,\mathrm{d}s = \mathrm{Tr}_P[r_1, s_1] \quad \text{for } r \text{ in } D \text{ and } s \text{ in } K. \tag{27}$$

That this is well-defined follows from Proposition A3, since

$$\mathrm{Res}_P r\,\mathrm{d}s = \mathrm{Tr}_p[(\mathrm{tr}(r))_1, s_1] = \mathrm{Res}_p \mathrm{tr}(r)\,\mathrm{d}s.$$

The latter also yields that the definitions of the residue map given before in this paper coincide with (27).

In order to compute $\mathrm{Res}_P r\,\mathrm{d}s$ for a given differential $r\,\mathrm{d}s$, one may proceed as follows: Let $B = \Lambda_p + s\Lambda_p$,

$$C = \{u \in B \mid ru \in \Lambda_p \text{ and } rsu \in \Lambda_p\}.$$

Then $\dim_k B/C$ is finite. Let Π be the k-linear projection of $\Lambda_p + r\Lambda_p + rs\Lambda_p$ onto Λ_p. Extend Π to a projection of D onto Λ_p. So ΠrD is a subset of Λ_p, i.e. Πr is in E_1. Therefore $\mathrm{Res}_P r\,\mathrm{d}s = \mathrm{Tr}_P[\Pi r, s]$. But $[\Pi r, s] = \Pi rs - s\Pi r$ maps D into B and C onto (0). Lemma A1 yields that $\mathrm{Tr}_P[\Pi r, s] = \mathrm{Tr}_{B/C}[\Pi r, s]$.

Lemma A4. *Let t be a generator for p in O_p, T a generator for P in Λ_p, and let $uT^e = t$, $e = e_p$ is the ramification index of p, u is a unit element in Λ_p. Then:*

(1) $\mathrm{Res}_P wT^n\,\mathrm{d}t = 0$ *for all units w in Λ_p and for all n in $\mathbb{Z}$, $n \neq -e$.*

(2) $\mathrm{Res}_P u^{-1}T^{-e}\,\mathrm{d}t = \dim_k \Lambda_p/T^e\Lambda_p$.

(3) *There exists an element v in $\Lambda_p - P$ such that* $\mathrm{Res}_P vT^{-e}\,\mathrm{d}t \neq 0$.

Proof. Applying the above we obtain

$$B = \Lambda_p + t\Lambda_p = \Lambda_p, \qquad C = \Lambda_p \cap T^{-n}\Lambda_p \cap T^{-n}t^{-1}\Lambda_p.$$

If $n \geq 0$, then $C = \Lambda_p$ and $\mathrm{Tr}_{B/C}$ is the zero map.

If $-e < -n < 0$, $C = T^n\Lambda_p$.

$$\mathrm{Res}_P wT^{-n}\,\mathrm{d}t = \mathrm{Tr}_{\Lambda_p/T^n\Lambda_p}(\Pi wT^{-n}t - t\Pi wT^{-n}).$$

Since $e > n$ and Π maps D onto Λ_p, the second term is in $T^n\Lambda_p$, so it has zero trace. But $wT^{-n}t = w'T^{e-n}$, Π being the identity on Λ_p yields $\mathrm{Res}_P wT^{-n}\,\mathrm{d}t = \mathrm{Tr}_{\Lambda_p/T^n\Lambda_p}(w'T^{e-n})$. The latter is zero since $w'T^{e-n}$ is nilpotent module $T^n\Lambda_p$.

If $-n < -e$, $C = T^m\Lambda_p$, where $m = \min(n, n-e)$.

$$\mathrm{Res}_P wT^{-n}\,\mathrm{d}t = \mathrm{Tr}_{\Lambda_p/T^m\Lambda_p}(\Pi wT^{-n}t).$$

But $wT^{-n}t$ is an element of $T^{-n}t\Lambda_p$, by definition Π maps $T^{-n}t\Lambda_p$ to zero, so the restriction of $\Pi wT^{-n}t$ to Λ_p is the zero map. Again $\mathrm{Res}_P wT^{-n}\,\mathrm{d}t = 0$ follows.

Finally if $-n = -e$, $C = T^e\Lambda_p$.

$$\mathrm{Res}_P wT^{-e}\,\mathrm{d}t = \mathrm{Tr}_{\Lambda_p/T^e\Lambda_p}(\Pi wT^{-e}t) = \mathrm{Tr}_{\Lambda_p/T^e\Lambda_p}(\Pi wu) = \mathrm{Tr}_{\Lambda_p/T^e\Lambda_p}(wu),$$

since wu is an element of Λ_p.

Because Tr is a non-degenerated linear form, there exists an element v' in $\Lambda_p - P$ such that $\mathrm{Tr}_{\Lambda_p/T^e\Lambda_p}(v') \neq 0$. So $\mathrm{Res}_P v'u^{-1} \neq 0$ which proves (3). If $w = u^{-1}$, we obtain

$$\mathrm{Res}_P u^{-1} T^{-e} \,\mathrm{d}t = \mathrm{Tr}_{\Lambda_p/T^e\Lambda_p}(1) = \dim_k \Lambda_p / T^e \Lambda_p. \qquad \square$$

Remark. (1) Part (3) of the above lemma may be used to solve the problem in the proof of Lemma 4.

(2) If $\hat{D}$ is complete with respect to a p-adic valuation, it is possible to extend Res_P to $\hat{D}$. The elements of $\hat{D}$ have a 'power series expansion' in T. It follows from Lemma A4, that the only term important to calculate the residue, is the term in T^{-e}.

In the classical approach to the commutative case this property is used to define the residue, cf. [3].

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